The Forcing Edge Chromatic Number of Some Standard Graphs

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ABSTRACT

Let S be a χ'-set of G. A subset $T \subseteq S$ is said to be a forcing subset for S if S is the unique χ'-set containing T. The forcing edge chromatic number $f_{\chi'}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The forcing edge chromatic number $f_{\chi'}(G)$ of G is the smallest forcing number of all χ'-sets of G. In this article, some general properties satisfied by this concept are studied and the forcing edge chromatic number of some standard graphs are determined. Also, connected graphs of order $n \geq 2$ edge chromatic number 0 or 1 or $\chi'(G)$ are characterized. It is shown that for a positive integer $a \geq 2$, there exists a connected graph G such that $f_{\chi'}(G) = \chi'(G) = a$.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite,undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be adjacent if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex.

A k-coloring of G is a function $c : V(G) \to \{1, 2, \ldots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. A p-vertex coloring of G is an assignment of p colors, $1,2, \ldots p$ to the vertices of G, the coloring is proper if no two distinct adjacent vertices have the same color. The minimum colours needed to colour the vertices of G is called *chromatic number* of G, denoted by $\chi(G)$.If $\chi(G) = p, G$ is said to be p - chromatic, where $p \leq k$. A set $C \subseteq V(G)$ is called *chromatic set* if C contains all vertices of distinct colors in G. The chromatic number of G is the minimum cardinality among all the chromatic sets of G. That is $\chi(G) = min\{|C_i| / C_i$ is a chromatic set of $G\}$. The concept of the chromatic number was studied in **[**2,3,4**]**. A k-edge coloring of G is a function $c' : E(G) \to \{1, 2, \ldots, k\}$, where $c'(e) \neq c'(f)$ for any two adjacent edges e and f in G. A p-edge coloring of G is an assignment of p colors, $1,2, \ldots p$ to the edges of G, the coloring is proper if no two distinct adjacent edges have the same color. The minimum colours needed to colour the edges of G is called *edge chromatic number* of G, denoted by $\chi'(G)$.If $\chi'(G) = p, G$ is said to be p *–edge chromatic*, where $p \leq k$. A set $C' \subseteq E(G)$ is called *edge chromatic set* if C' contains all edges of distinct colors in G. The *egde chromatic number* of G is the minimum cardinality among all the edge chromatic sets of G. That is $\chi'(G) = min\{|C_i'|/ C_i' \text{is a edge chromatic set of } G\}$. An edge chromatic set of cardinality $\chi'(G)$ is called a χ'-set of G.The edge-chromatic number $\chi'(G)$ of G is defined to be the least number of colours needed to colour the edges of G in such a way that no two adjacent edges have the same colour. The concept of the edge chromatic number was studied in **[**5,6,7**]**. The chromatic number has application in Time Table Scheduling, Map coloring, channel assignment problem in radio technology, town planning, GSM mobile phone networks etc.**[**8,9**]**.

2. The forcing edge chromatic number of some standard graphs

Definition 2.1. Let S be a χ'-set of G. A subset $T \subseteq S$ is said to be a forcing *subset* for$S$if S is the unique χ'-set containing T. The *forcing edge chromatic number* $f_{\chi'}(S)$ of Sin $G$is the minimum cardinality of a forcing subset for S. The forcing edge chromatic number $f_{\chi'}(G)$ of G is the smallest forcing number of all χ'-sets of G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{e_1, e_2, e_3, e_4\}$,$S_2 = \{e_6, e_2, e_3, e_4\}$, $S_3 = \{e_1, e_5, e_3, e_4\}$,$S_4 = \{e_6, e_5, e_3, e_4\}$ are the only two χ'-sets of G such that $\chi'(G) = 3$, $f_{\chi'}(S_1) = f_{\chi'}(S_2) = f_{\chi'}(S_3) = f_{\chi'}(S_4) = 2$ so that $f_{\chi'}(G) = 2$.

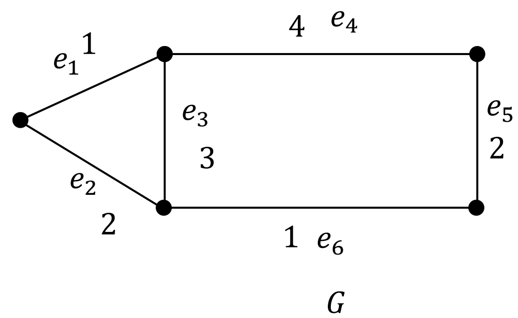

Figure 2.1

The following result follows immediately from the definitions of the edge chromatic number and the forcing edge chromatic number of a connected graph G.

Observation 2.3. For every connected graph$G$, $0 \le f_{\chi'}(G) \le \chi'(G)$.

Remark 2.4. The bounds in the Observation 2.3 are sharp. For the complete graph $G = K_3$, $S = E(G)$ is the unique χ'-set of G so that $f_{\chi'}(G) = 0$. For the graph G given in Figure 2.2, $S_1 = \{e_1, e_2, e_3\}$, $S_2 = \{e_1, e_4, e_5\}$, $S_3 = \{e_1, e_2, e_6\}$, $S_4 = \{e_1, e_4, e_6\}$, $S_5 = \{e_3, e_2, e_5\}$, $S_6 = \{e_3, e_4, e_5\}$, $S_7 = \{e_3, e_2, e_6\}$, $S_8 = \{e_3, e_4, e_6\}$such that$f_{\chi'}(S_i) = 3$ for $i = 1$

to 8 and $\chi'(G)$ = 3so that $f_{\chi'}(G) = \chi'(G) = 3$. Also the bounds are strict. For the graph in Figure 2.1, $\chi'(G) = 4, f_{\chi'}(G) = 2$. Thus $0 < f_{\chi'}(G) < \chi'(G)$.

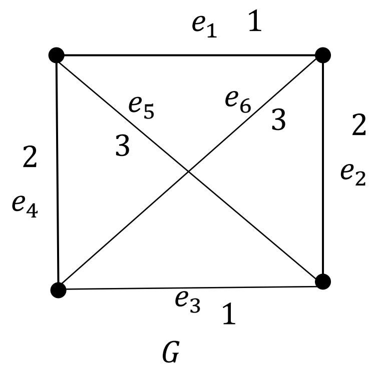

Figure 2.2

Definition: 2.5. An edge e of a graph G is said to be an *edge chromatic edge* of G if e belongs to every χ'-set of G.

Example 2.6. For the graph G given in Figure 2.3, $S_1 = \{e_1, e_2, e_3\}, S_2 = \{e_1, e_2, e_4\}$, $S_3 = \{e_5, e_2, e_3\}, S_4 = \{e_5, e_2, e_4\}$ are the only χ'-sets of G such that e_2 is a chromatic edge of G.

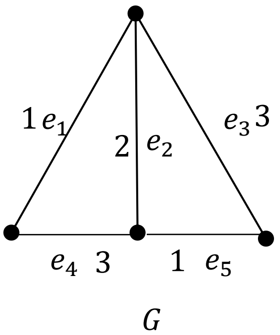

Figure 2.3

Theorem 2.7. Let G be a connected graph of order $n \geq 2$ with $\Delta(G) = n - 1$. Let x be auniversal vertex of G and e be an edge incident with x. Then e is a chromatic edge of G.

Proof. On the contrary, suppose that e is not a chromatic edge of G. Then there exists a χ'-set S of G such that$e = uv$. Let $c(e) = c_1$. Since$e \notin S$, there exists $f = yz \in E(G)$ such that $c(f) = c_1$ and $y \neq x, v$ and $z \neq x, v$. Hence it follows that x is not a universal vertex of G, which is a contradiction. Therefore e is a chromatic edge of G.

Theorem 2.8. Let G be a connected graph. Then

(a) $f_{\chi'}(G) = 0$ if and only if G has a uniqueχ'-set.

(b) $f_{\chi'}(G) = 1$ if and only if G has at least twoχ'-sets, one of which is a unique χ'-set containing one of its elements, and

(c) $f_{\chi'}(G) = \chi'(G)$if and only if no χ' -set of G is the unique χ'–set containing any of its proper subsets.

Proof. (a) Let $f_{\chi'}(G) = 0$. Then, by definition, $f_{\chi'}(S) = 0$ for some χ' -set S of G so that the empty set $\phi$is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique χ'-set of G. The converse is clear.

(b) Let $f_{\chi'}(G) = 1$. Then by Theorem 2.8(a), G has at least two χ'-sets. Also, since $f_{\chi'}(G) = 1$, there is a singleton subset T of a χ'-set S of G such that T is not a subset of any other χ' -set of G. Thus S is the unique χ' -set containing one of its elements. The converse is clear.

(c) Let $f_{\chi'}(G) = \chi'(G)$. Then $f_{\chi'}(G) = \chi'(G)$for every χ'-set S in G. Also, by Observation 2.3, $\chi'(G) \geq 2$ and hence $f_{\chi'}(G) \geq 2$. Then by Theorem 2.8(a), G has at least two χ'-sets and so the empty set ϕ is not a forcing subset for any χ'-set of G. Since $f_{\chi'}(S) = \chi'(G)$, no proper subset of S is a forcing subset of S. Thus no χ'-set of G is the unique χ'-set containing any of its proper subsets. Conversely, the data implies that G contains more than one χ' -set and no subset of any χ' -set S other than S is a forcing subset for S. Hence it follows that $f_{\chi'}(G) = \chi'(G)$.

Theorem 2.9. Let G be a connected graph and W be the set of all chromaticedges of G. Then$f_{\chi'}(G) \leq \chi'(G) - |W|$.

Proof. Let S be any χ'-set of G. Then $\chi'(G) = |S|$, $W \subseteq S$ and S is the unique χ'-set containing $S - W$. Thus $f_{\chi'}(G) \leq |S - W| = |S| - |W| = \chi'(G) - |W|$.

In the following we determine the forcing edge chromatic number of some standard graphs.

Theorem 2.10. For the complete graph $G = K_n (n \geq 2)$,

$$f_{\chi'}(G) = \begin{cases} 0 & if\ n = 2,3 \\ 3 & if\ n = 4 \\ n-1 & if\ n \geq 5 \end{cases}$$

Proof. For $n = 2$ and $n = 3$, $S = E(G)$ is the unique χ'-set of G, the result follows from Theorem 2.8 (a). For $n = 4$, let $e_{11} = v_1v_2$, $e_{12} = v_1v_3$, $e_{13} = v_1v_4$, $e_{21} = v_2v_3$, $e_{22} = v_2v_4$, $e_{31} = v_3v_4$. Assign $c'(e_{11}) = c'(e_{31}) = 1$, $c'(e_{12}) = c'(e_{22}) = 2$, $c'(e_{13}) = c'(e_{21}) = 3$. Then $S_1 = \{e_{11}, e_{12}, e_{13}\}$, $S_2 = \{e_{11}, e_{12}, e_{21}\}$, $S_3 = \{e_{11}, e_{22}, e_{13}\}$, $S_4 = \{e_{11}, e_{22}, e_{21}\}$, $S_5 = \{e_{31}, e_{12}, e_{13}\}$, $S_6 = \{e_{31}, e_{12}, e_{21}\}$, $S_7 = \{e_{31}, e_{22}, e_{13}\}$, $S_8 =$

$\{e_{31}, e_{22}, e_{21}\}$ are the χ'-set of G such that $f_{\chi'}(S_i) = 3$ for $i = 1$ to 8 so that $f_{\chi'}(G) = 3$. For $n \geq 5$, let $e_{1j} = v_1 v_j (2 \leq j \leq n), e_{2j} = v_2 v_j (3 \leq j \leq n)$, $e_{3j} = v_3 v_j (4 \leq j \leq n)$, $\ldots, e_{(n-1)j} = v_{n-1} v_n$. Assign $c'(e_{1j}) = c'_j$, $c'(e_{2j}) = c'_j - 1\ (1 \leq j \leq n-1)$, $c'(e_{3j}) = c'_j - 2\ (1 \leq j \leq n-1), \ldots\ldots\ldots, c'(e_{(n-1)j}) = c'_j - (n-2)\ (1 \leq j \leq n-1)$,
$c'(e_{(n-2)j}) = n$ so that $\chi'(G) = n$. Since $e_{(n-2)j}$ is a chromatic edge of G, by Theorem 2.9, $f_{\chi'}(G) \leq n-1$. Let S be a chromatic edge set of G. We prove that $f_{\chi'}(G) = n-1$. On the contrary, suppose that $f_{\chi'}(G) \leq n-2$. Then there exists a forcing subset T of S such that $|T| \leq n-2$. Let $e \in S$ such that $e \notin T$. Then e is not a chromatic edge of G. Without loss of generality, let us assume that $c'(e) = c'_1$. Since $n \geq 5$, there exists $f \in E(G)$ such that $c'(f) = c'_1$. Let $S' = [S - \{e\}] \cup \{f\}$. Then S' is a χ'-set of G. Hence T is a proper subset of a χ'-setS' of G, which is a contradiction. Therefore $f_{\chi'}(G) = n-1$.

Theorem 2.11. For the star graph $G = K_{1,n-1}$ $(n \geq 3)$, $f_{\chi'}(G) = 1$.

Proof. Since $S = E(G)$ is the unique χ'-set of G, the result follows from Theorem 2.8(a)

Theorem 2.12. For the double star graph $G = K_{2,r,s}$, $f_{\chi'}(G) = 2$.

Proof. Let $V = \{x, v_1, v_2, \ldots, v_r\} \cup \{y, u_1, u_2, \ldots, u_s\}$ be the vertex set of G. Let $f_i = xv_i$, $e = xy, g_i = yu_j$ be the edge set of G for all $(1 \leq i \leq r)$ and $(1 \leq j \leq s)$where $r + s = n - 2$. Then $S_1 = \{e, f_i\}$ $(1 \leq i \leq r)$ and $S_2 = \{e, g_j\}(1 \leq j \leq s)$ are the only χ' -sets of G such that $f_{\chi'}(S_1) = f_{\chi'}(S_2) = 2$ so that$f_{\chi'}(G) = 2$.

Theorem 2.13. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$,

$$f_{\chi'}(G) = \begin{cases} 0 & if\ r = 1, s = 1 \\ 2 & if\ r = 2, s = 2 \\ s & if\ 2 \leq r \leq s \end{cases}$$

Proof. For $r = 1$ and $s \geq 2$, then the result follows from Theorem 2.11. For $r = 2$ and $s = 2$, $S_1 = \{e_{11}, e_{12}\}$,$S_2 = \{e_{11}, e_{21}\}$,$S_3 = \{e_{22}, e_{12}\}$, $S_4 = \{e_{22}, e_{21}\}$ are the χ'-sets of G such that $f_{\chi'}(S_i) = 2$ for $i = 1$ to 4 so that $f_{\chi'}(G) = 2$. So let $2 \leq r \leq s$. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the bipartite sets of G.Let $e_{1j} = x_1 y_j (1 \leq j \leq s), e_{2j} = x_2 y_j (1 \leq j \leq s)$, $\ldots\ldots\ldots, e_{ij} = x_i y_j (1 \leq i \leq r)(1 \leq j \leq s)$. Assign $c'(e_{1j}) = c'_j (1 \leq j \leq s), c'(e_{2j}) = c'_j, c'_1 (2 \leq j \leq s), c'(e_{3j}) = c'_j, c'_1, c'_2 (3 \leq j \leq), \ldots\ldots\ldots, c'(e_{ij}) = c'_s, c'_1, c'_2, \ldots\ldots\ldots, c'_{s-1} (1 \leq i \leq r)(1 \leq j \leq s)$. Then $S_{ij} = \{e_{11}, e_{12}, e_{13}, \ldots, e_{ks}\}$ is a χ'-set of G such that $\chi'(G) = s$. By Observation 2.3, $0 \leq f_{\chi'}(G) \leq s$. Since χ'-set of G is not unique $f_{\chi'}(G) \geq 1$. It is easily observed that no singleton subsets or two element subsets of

$S_{ij}(1 \le i \le r), (1 \le j \le s)$ is not a forcing subset of S_{ij} so that $f_{\chi'}(S_{ij}) = s$. Since this true for all χ'-set $S_{ij}(1 \le i \le r), (1 \le j \le s)$ so that $f_{\chi'}(G) = s$.

Theorem 2.14. For the path $G = P_n$ $(n \ge 3)$, $f_{\chi'}(G) = \begin{cases} 0 & if\ n = 3 \\ 1 & if\ n = 4 \\ 2 & if\ n \ge 5 \end{cases}$

Proof. Let P_n be $v_1, v_2, \dots, v_n$ and let $e_i = v_i v_{i+1}(1 \le i \le n-1)$. For $n = 3$, $S = E(G)$ is the unique χ'-set of G, the result follows from Theorem 2.8(a). For $n = 4, S_1 = \{e_1, e_2\}$ and $S_2 = \{e_2, e_3\}$ are the χ'-sets of G such that $f_{\chi'}(S_1) = f_{\chi'}(S_2) = 1$ so that $f_{\chi'}(G) = 1$. So let $n \ge 5$. Then $S_i = \{e_i, e_{i+1}\}(1 \le i \le n-1)$ and $S_{jk} = \{e_j, e_k\}(1 \le j \le k \le n-1)$ and $|j-k|$is odd are the only χ'-sets of G such that $f_{\chi'}(S_i) = 2$ for $(1 \le i \le n-1)$ and $f_{\chi'}(S_{jk}) = 2$ for $(1 \le j \le k \le n-1)$ so that$f_{\chi'}(G) = 2$.

Theorem 2.15. For the cycle $G = C_n$ $(n \ge 4)$, $f_{\chi'}(G) = 2$.

Proof. Let C_n be $v_1, v_2, \dots, v_n, v_1$ and let $e_i = v_i v_{i+1}(1 \le i \le n-1)$ and $e_n = v_n v_1$. We consider the following two cases.

Case(1) n is even

$$c(e_i) = \begin{cases} 1, & if\ i\ is\ odd \\ 2, & if\ i\ is\ even \end{cases}$$

Then $S_i = \{e_i, e_{i+1}\}(1 \le i \le n-1)$ and $S_{ijk} = \{e_j, e_k\}$ $(1 \le j \le k \le n-1)$ and $|j-k|$is odd are the only χ'-sets of G such that $f_{\chi'}(S_i) = 2$ and $f_{\chi'}(S_{jk}) = 2$ for $(1 \le i \le n-1)$ and $(1 \le j \le k \le n-1)$ so that $f_{\chi'}(G) = 2$.

Case(2) n is odd

$$c(e_i) = \begin{cases} 1 & if\ n\ \ is\ odd \\ 2 & if\ n\ \ is\ even \\ 3 & if\ i = n \end{cases}$$

Since $E(v_n v_1)$ is the set of chromatic edges of $G, E(v_n v_1)$ is a subset of every χ'-set of G. It can be easily seen that any χ'-set of G is of the form $S = E(v_n v_1) \cup \{x, y\}$, where $x, y \in \{e_1, e_2, \dots, e_{n-1}\}$ so that $\chi'(G) = n + 2$. By Theorem 2.9, $f_{\chi'}(G) \le n + 2 - n = 2$. Since χ'-set of G is not unique $f_{\chi'}(G) \ge 1$. It is easily observed that no singleton subsets of S is not a forcing subset of S so that $f_{\chi'}(S) = 2$. Since this is true for all χ'-set S of G, $f_{\chi'}(G) = 2$.

Theorem 2.16. For a positive integer $a \ge 2$, there exists a connected graph G such that $f_{\chi'}(G) = \chi'(G) = a$.

Proof. For $a = 2$, let $G = C_4$. Then by Theorem 2.15, $f_{\chi'}(G) = \chi'(G) = a$. So, let

$a \geq 3$. Let $G = K_{2,a}$. By Theorem 2.13, $f_{\chi'}(G) = \chi'(G) = a$.

3. Conclusion

In this article, we discuss about a new concept namely, forcing edge chromatic number of a graph. Also, the relation between edge chromatic number and forcing edge chromatic number is found. The above concept is examined by some standard graphs with examples.

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